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REDUCTION OF VARIABLES FOR WILLMORE-CHEN SUBMANIFOLDS IN SEVEN SPHERES

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ABSTRACT

We obtain a reduction of variables criterion for 4-dimensional Willmore–Chen submanifolds associated with the generalized Kaluza–Klein conformal structures on the 7-sphere. This argument connects the variational problem of Willmore–Chen with a variational problem for closed curves into 4-spheres. It involves an elastic energy functional with potential. The method is based on the extrinsic conformal invariance of the Willmore–Chen variational problem, and the principle of symmetric criticality. It also uses several techniques from the theory of pseudo-Riemannian submersions. Furthermore, we give some applications, in particular, a result of existence for constant mean curvature Willmore–Chen submanifolds which is essentially supported on the nice geometry of closed helices in the standard 3-sphere.

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1. Introduction

Willmore–Chen submanifolds are the natural extension to higher dimensions of Willmore surfaces. They are critical points of the Willmore–Chen functional

$$\mathcal{W}(M) = \int_M (\alpha^2 - \tau_e)^{\frac{n}{2}} dV,$$

defined on the space of compact n-dimensional submanifolds M of a given Riemannian (perhaps pseudo-Riemannian, if the submanifolds are choosen to be nondegenerate) manifold $\bar{M}$. The terms appearing in the integrand are the mean curvature function α and the extrinsic scalar curvature τ_e of M into $\bar{M}$ (τ_e is the difference between the scalar curvature of M and the scalar curvature of M when computed with the ambient Riemannian curvature).

The importance of the variational problem associated with this functional partially comes from its invariance under conformal changes of the metric of the ambient space $\bar{M}$ (see [10]). Therefore, $\mathcal{W}$ is also called the **conformal total mean curvature** functional. When $n = 2$, $\mathcal{W}$ is the well-known Willmore functional, and its critical points are the Willmore surfaces. Minimal surfaces in standard spheres are trivial examples of Willmore surfaces. However, some papers showing several methods to obtain non-minimal Willmore surfaces in standard spheres are known in the literature (see [3, 7, 11, 16, 20], etc.), or in non-standard spheres (see [1, 8]) and, more recently, in spaces with a *global warped product* pseudo-Riemannian structure (see [2, 4]). On the other hand, the first non-trivial examples of Willmore–Chen submanifolds in standard spheres were given in [6] and later in [4] for conformal structures associated to warped product metrics and consequently on reducible spaces.

In this paper, we deal with the Willmore–Chen submanifolds of the conformal classes on the 7-sphere associated with the so-called *generalized Kaluza–Klein (Riemannian even pseudo-Riemannian) metrics.* These are *local warped product metrics* ([13, 21]), and include, as particular cases, the Kaluza–Klein (also called **bundle-like**) metrics. An essential ingredient to construct these structures is the usual Hopf fibration $\pi\colon \mathbb{S}^7 \longrightarrow \mathbb{S}^4$, which can be regarded as a Riemannian submersion when both spheres are assumed to be round spheres of radii, for example, 2 and 1, respectively. Therefore, the canonical variation of this Riemannian submersion ([9]) gives a one-parameter family of bundle-like metrics on $\mathbb{S}^7$, which was earlier used to find examples of homogeneous Einstein metrics. For example, in contrast with the case of the Hopf fibration $p\colon \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ which cannot admit any Einstenian metric on $\mathbb{S}^3$ different to the standard (because of its dimension), the case $\pi\colon \mathbb{S}^7 \longrightarrow \mathbb{S}^4$ gives exactly one Einstenian metric, different to the standard one. Of course, in both cases all the metrics in the canonical variation have constant scalar curvature.

A generalized Kaluza–Klein metric on $\mathbb{S}^7$ is rich in isometries. In fact, the natural action of $\mathbb{S}^3$ on $\mathbb{S}^7$, to give $\mathbb{S}^4$ as space of orbits, is made up of isometries of any generalized Kaluza–Klein metric on $\mathbb{S}^7$. Furthermore, 4-dimensional $\mathbb{S}^3$-invariant submanifolds on $\mathbb{S}^7$ correspond with complete liftings of curves into the base $\mathbb{S}^4$.

The main theorem of this paper, which is contained in section 4, is, in the sense of Palais ([19]), an example of reduction of variables for a variational problem. It can be explained as follows: the search of Willmore–Chen submanifolds, for generalized Kaluza–Klein structures on $\mathbb{S}^7$, which do not break the $\mathbb{S}^3$-symmetry, is reduced via the principle of symmetric criticality to the search of closed curves into the base $\mathbb{S}^4$ which are critical points of a functional of the type

$$\mathcal{G}(\gamma) = \int_\gamma (\kappa^2 + \phi(\gamma'))^2 ds,$$

where κ denotes the curvature function of the closed curve γ for a certain metric on $\mathbb{S}^4$ and ϕ is a potential defined on the corresponding unit tangent vector bundle.

We also compute the Euler–Lagrange equations associated to $\mathcal{G}$, when the potential ϕ is defined on $\mathbb{S}^4$, and obtain some simple consequences.

In the last section, we take advantage of the geometry of helices in a round 3-sphere and use the fact that the critical points of $\mathcal{G}$, when ϕ is constant, in a round 4-sphere must be contained in a totally geodesic 3-sphere (see Proposition 3). Then, we construct in wide families of conformal structures on $\mathbb{S}^7$ a rational one-parameter family of Willmore–Chen submanifolds with constant mean curvature (see Corollaries 2, 3 and Remark 5).

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2. Set up

We will use the terminology of [9, 18]. Let π: $(P, \tilde{g}) \longrightarrow (B, \check{g})$ be a pseudo-Riemannian submersion. For each $x \in P$ we denote by $F_x = \pi^{-1}(\pi(x))$ the fibre (or leaf) through x. Then $\vartheta_x = T_x(F_x)$ defines the vertical distribution, and the orthogonal complement to ϑ_x in T_xP, say $\mathcal{H}_x$, defines the horizontal distribution $\mathcal{H}$. The O'Neill invariants will be denoted by T and A. Recall that T (also called the vertical configuration tensor [13]) is defined from the second fundamental form of the fibres. In particular, it vanishes identically if and only if the fibres are totally geodesic. On the other hand, the horizontal configuration tensor A measures the obstruction to integrability of the horizontal distribution.

Given an arclength parametrized immersed curve γ into B, we consider the submanifold $M_\gamma = \pi^{-1}(\gamma)$ on P. If $X = \gamma'$ and $\bar{X}$ is its horizontal lift to P, then $T_x(M_\gamma) = \mathrm{Span}\{\bar{X}(x), \vartheta_x\}$ and hence the normal space $T^\perp(M_\gamma)$ is a horizontal subspace. Pseudo-Riemannian submersions with minimal submanifolds as fibres are harmonic as maps.

For a pseudo-Riemannian submersion one can define a nice deformation of the metric $\tilde{g}$ by changing the relative scales of the base and the fibres. In fact we just define $\{\tilde{g}^t = \pi^*(\breve{g}) + t^2 g_F \quad /t > 0\}$, where g_F denotes the induced metric of $\tilde{g}$ on the fibres. Certainly we obtain a one-parameter family of pseudo-Riemannian submersions $\pi_t\colon (P, \tilde{g}^t) \longrightarrow (B, \breve{g})$ with the same horizontal distribution.

Along this paper, the Hopf maps between spheres will be important. We will denote then $p\colon \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ and $\pi\colon \mathbb{S}^7 \longrightarrow \mathbb{S}^4$. Recall that they are submersions with fibre $\mathbb{S}^1$ and $\mathbb{S}^3$, respectively. We are going to describe briefly the second one. Similar facts hold for the first one. From now on, $\mathbb{S}^n(r)$ will denote the round n-sphere of radius r, which means it is endowed with its standard metric of constant sectional curvature $1/r^2$.

We start with $\mathbb{H}^2 = \{y = (y_1, y_2)/y_1, y_2 \in \mathbb{H}\}$, where $\mathbb{H}$ denotes the algebra of the quaternions equipped with its natural symplectic product $(,)$ whose real part gives the usual Euclidean inner product on $\mathbb{R}^8 = \mathbb{H}^2$, which will be denoted by $\langle,\rangle$. Then $\mathbb{S}^7(r) = \{y \in \mathbb{H}^2/(y,y) = r^2\}$ and its tangent space is $T_x\mathbb{S}^7(r) = \{y \in \mathbb{H}^2/\langle y, x\rangle = 0\}$.

We have an involutive distribution on $\mathbb{S}^7(r)$ defined by

$$x \longrightarrow \vartheta_x = \mathrm{Span}\{ix, jx, kx\},$$

where i, j and k are units of $\mathbb{H}$. The space of orbits, under the natural action of $\mathbb{S}^3$ (or $\mathbb{S}^3(1)$, i.e., the unit quaternions) on $\mathbb{S}^7(r)$, is the quaternion projective line, which can be identified with the 4-sphere. Now, the quaternion Hopf mapping $\pi\colon \mathbb{S}^7(r) \longrightarrow \mathbb{S}^4(\frac{1}{2}r)$ is a Riemannian submersion whose fibres (i.e., the orbits) are totally geodesic submanifolds in $\mathbb{S}^7(r)$ and hence they have the isometry type of $\mathbb{S}^3(r)$.

As usual we denote by overbars the horizontal lifts of corresponding objects on the base. Thus, given a unit speed curve γ in $\mathbb{S}^4(\frac{1}{2}r)$, we can talk about the horizontal lifts $\bar{\gamma}$ of γ to $\mathbb{S}^7(r)$, which are unit speed curves on $\mathbb{S}^7(r)$. The set of those lifts defines the complete lift $M_\gamma = \pi^{-1}(\gamma)$ of γ.

If $\bar{D}$ and D denote the Levi-Civita connections of $\mathbb{S}^7(r)$ and $\mathbb{S}^4(\frac{1}{2}r)$, respectively, then we have

$$\bar{D}_{\bar{X}}\bar{Y} = \overline{D_X Y} - \langle i\bar{X}, \bar{Y}\rangle V_1 - \langle j\bar{X}, \bar{Y}\rangle V_2 - \langle k\bar{X}, \bar{Y}\rangle V_3, \tag{2.1}$$

$$\begin{cases} \bar{D}_{\bar{X}} V_1 = \bar{D}_{V_1}\bar{X} = (1/r) i\bar{X}, \\ \bar{D}_{\bar{X}} V_2 = \bar{D}_{V_2}\bar{X} = (1/r) j\bar{X}, \\ \bar{D}_{\bar{X}} V_3 = \bar{D}_{V_3}\bar{X} = (1/r) k\bar{X}, \end{cases} \tag{2.2}$$

where

$$V_1(x) = \frac{1}{r} ix, \quad V_2(x) = \frac{1}{r} jx \quad \text{and} \quad V_3(x) = \frac{1}{r} kx.$$

3. Generalized Kaluza–Kein pseudo-metrics in the seven-sphere

Let $\mathbb{S}^7(\mathbb{S}^4, \mathbb{S}^3)$ be the principal fibre bundle with base $\mathbb{S}^4$ and structure group $\mathbb{S}^3$. The projection map is $\pi: \mathbb{S}^7 \longrightarrow \mathbb{S}^4$ and $\mathcal{L}$ will denote the Lie algebra of $\mathbb{S}^3$. A natural connection can be defined on this bundle by assigning to each $x \in \mathbb{S}^7$ its horizontal subspace $\mathcal{H}_x$. Let ω be the connection 1-form, which is defined on S^7 with values into $\mathcal{L}$. In particular, $V_1(x) = ix$, $V_2(x) = jx$ and $V_3(x) = kx$ give a global frame of fundamental vector fields. For the sake of simplicity we assume both $\mathbb{S}^7$ and $\mathbb{S}^3$ to have radius one and so $\{V_1, V_2, V_3\}$ defines an orthonormal frame with respect to the standard Riemannian metric $d\sigma^2$ of $\mathbb{S}^3$. This metric is known to be bi-invariant on this Lie group.

Let $\mathcal{M}$ and $\mathcal{F}_+$ be the space of Riemannian metrics and the space of smooth positive functions on $\mathbb{S}^4$, respectively. We define the following mapping:

$$\Phi: \mathcal{M} \times \mathcal{F}_+ \times \{-1, 1\} \longrightarrow \bar{\mathcal{M}},$$

$$\bar{h} = \Phi(h, v, \epsilon) = \pi^*(h) + \epsilon(v \circ \pi)^2 \omega^*(d\sigma^2), \tag{3.1}$$

where $\bar{\mathcal{M}}$ is the space of pseudo-metrics on $\mathbb{S}^7$. Thus $\bar{h}$ is Riemannian or index three according to whether ϵ is 1 or -1.

A pseudo-metric on S^7 in the image of this mapping is called a **generalized Kaluza–Klein metric** on $\mathbb{S}^7(\mathbb{S}^4, \mathbb{S}^3)$, or shortly, on $\mathbb{S}^7$. The Kaluza–Klein metric (also known as **bundle-like metric** in the literature) corresponds with the case where v is a constant. In particular, when $h = g$ (the standard metric of constant sectional curvature 4 on $\mathbb{S}^4$), then $\Phi(g, t, 1)$ gives the canonical variation of the standard Riemannian metric $\bar{g} = \Phi(g, 1, 1)$ (of constant sectional curvature 1) on $\mathbb{S}^7$. Furthermore, $\Phi(g,t,-1)$ defines the canonical variation of the standard index three metric $\bar{g}_3 = \Phi(g,1,-1)$ on $\mathbb{S}^7$.

Remark 1: It should be noticed that $\bar{g}_3$ is essentially obtained from $\bar{g}$ by changing the sign of $d\sigma^2$. In greater generality, we can consider the standard Lorentz metric $d\sigma_1^2$ on $\mathbb{S}^3$ (see [23] for details about the standard metric on an odd sphere) and construct the associated generalized Kaluza–Klein metrics by putting $d\sigma_1^2$ instead of $d\sigma^2$ into (3.1). In this case $\Phi(g,t,1)$ gives the canonical variation of the standard Lorentz metric $\bar{g}_1 = \Phi(g,1,1)$ on $\mathbb{S}^7$, while $\Phi(g,t,-1)$ defines the canonical variation of the index two standard metric $\bar{g}_2 = \Phi(g,1,-1)$ on $\mathbb{S}^7$.

It can be summarized in a few words: *generalized Kaluza–Klein metrics are local warped product metrics* ([9, 13]).

Some nice properties of these metrics are collected in the following

PROPOSITION 1: *Let $\bar{h} = \Phi(h,v,\epsilon)$ be a generalized Kaluza–Klein metric on $\mathbb{S}^7$. The following assertions hold:*

(1) *$\pi\colon (\mathbb{S}^7,\bar{h}) \longrightarrow (\mathbb{S}^4,h)$ is a pseudo-Riemannian submersion. Moreover, it has totally geodesic leaves if and only if v is a constant.*
(2) *The natural action of $\mathbb{S}^3$ on $\mathbb{S}^7$ is made by isometries of $(\mathbb{S}^7,\bar{h})$.*
(3) *An immersed 4-dimensional submanifold M in $\mathbb{S}^7$ is $\mathbb{S}^3$-invariant if and only if $M = M_\gamma = \pi^{-1}(\gamma)$ for some immersed curve γ into $\mathbb{S}^4$. If γ is closed, then M is compact and it is embedded in $\mathbb{S}^7$ if γ does not have self-intersections in $\mathbb{S}^4$.*
(4) *If v is constant, then the mean curvature function α of M_γ in $(\mathbb{S}^7,\bar{h})$ and the curvature function κ of γ in $(\mathbb{S}^4,h)$ are related by*

$$\alpha^2 = \frac{1}{16}(\kappa^2 \circ \pi). \tag{3.2}$$

Proof: The first claim is evident. Moreover, the O'Neill invariant T ([18]) of these pseudo-Riemannian submersions vanishes identically if and only if v is a constant. This proves (1). The second statement is clear, and the third one can be obtained by using the argument given in [20]. The last assertion is a particular case of a result shown in [1].

4. A general approach. The main theorem

Let M be a compact smooth manifold of dimension 4 and denote by $I(M,\mathbb{S}^7)$ the space of immersions of M into $\mathbb{S}^7$. For any pseudo-Riemannian metric $\tilde{h}$ on $\mathbb{S}^7$, we define the submanifold $I_{\tilde{h}}(M,\mathbb{S}^7)$ of $I(M,\mathbb{S}^7)$ by $I_{\tilde{h}}(M,\mathbb{S}^7) =$

$\{\varphi \in I(M,\mathbb{S}^7)/\varphi^*(\tilde{h})$ is non-degenerate$\}$. Then the Willmore–Chen functional is $\mathcal{W}: I_{\tilde{h}}(M,\mathbb{S}^7) \longrightarrow \mathbb{R}$, which is given by

$$\mathcal{W}(\varphi) = \int_M (\alpha^2 - \tau_e)^2 dV. \tag{4.1}$$

To study Willmore–Chen submanifolds in $(\mathbb{S}^7, \mathcal{C}(\bar{h}))$ (where $\mathcal{C}(\bar{h})$ denotes the conformal structure on S^7 associated with $\bar{h}$), we make the following conformal change in $(\mathbb{S}^7, \bar{h})$:

$$\tilde{h} = \frac{1}{(\pi \circ v)^2}\bar{h} = \pi^*\Big(\frac{1}{v^2} \cdot h\Big) + \epsilon \cdot \omega^*(d\sigma^2). \tag{4.2}$$

Now $\pi: (\mathbb{S}^7, \tilde{h}) \longrightarrow (\mathbb{S}^4, \frac{1}{v^2} \cdot h)$ provides us with the following facts:

1. It is still a pseudo-Riemannian submersion. However, it has totally geodesic leaves.
2. $I_{\tilde{h}}(M,\mathbb{S}^7) = I_{\bar{h}}(M,\mathbb{S}^7)$ and φ is Willmore–Chen in $(\mathbb{S}^7, \tilde{h})$ if and only if it is so in $(\mathbb{S}^7, \bar{h})$.
3. For any $\varphi \in I_{\tilde{h}}(M,\mathbb{S}^7)$ and any $a \in \mathbb{S}^3$, we have $\mathcal{W}(\varphi \cdot a) = \mathcal{W}(\varphi)$. In other words, $\mathcal{W}$ is $\mathbb{S}^3$-invariant.

From now on we will adopt the following notation: $G = S^3$, J_G is the space of G-invariant points of $I_{\tilde{h}}(M,\mathbb{S}^7)$, Σ is the set of critical points of $\mathcal{W}$ (that is, the set of Willmore–Chen immersions) and Σ_G is the set of critical points of $\mathcal{W}$ when restricted to J_G. As a consequence of statement (3) in Proposition 1, J_G can be identified with $\{M_\gamma = \pi^{-1}(\gamma)/\gamma$ is a closed curve inmersed into $\mathbb{S}^4\}$. On the other hand, the principle of symmetric criticality ensures ([19]),

$$\Sigma \cap J_G = \Sigma_G.$$

Therefore, in order to obtain Willmore–Chen submanifolds which do not break the G-symmetry of the problem, we only need to compute $\mathcal{W}$ on J_G and then to proceed in due course. To compute the first term of the integrand of $\mathcal{W}(M_\gamma)$, we use (3.2), where κ is the curvature function of γ into $(\mathbb{S}^4, \frac{1}{v^2 \cdot h})$. We choose an arclength parametrization of γ in $(\mathbb{S}^4, \frac{1}{v^2 \cdot h})$. So if $L > 0$ is the length of γ and $\bar{\gamma}$ is a horizontal lift of γ to $(\mathbb{S}^7, \tilde{h})$, we define a mapping

$$\Psi: [0,L] \times \mathbb{S}^3 \longrightarrow M_\gamma, \quad \Psi(s,a) = \bar{\gamma}(s) \cdot a.$$

It is clear that Ψ extended to $\mathbb{R} \times S^3$ defines a covering, which can be used to obtain parametrizations, of M_γ. For the sake of simplicity we let V_0 denote the horizontal lift of γ' to $(\mathbb{S}^7, \tilde{h})$ along M_γ. Then $\{V_0, V_1, V_2, V_3\}$ provides a global frame of unit vector fields on M_γ.

The extrinsic scalar curvature τ_e of M_γ is given by

$$\tau_e = \frac{1}{12} \sum_{l,m=0}^{3} (K(V_l, V_m) - \tilde{K}(V_l, V_m)),$$

where $\tilde{K}$ and K denote the sectional curvature of $(\mathbb{S}^7, \tilde{h})$ and M_γ, respectively. We use the local Riemannian product structure of M_γ together with the totally geodesic nature of the leaves into both M_γ and $(\mathbb{S}^7, \tilde{h})$ to obtain

$$\tau_e = -\frac{1}{6} \sum_{l=1}^{3} \tilde{K}(V_0, V_l). \tag{4.3}$$

The O'Neill invariant A (also called the horizontal configuration map, [18, 13]) allows one to compute the mixed (also called vertizontal, [22]) sectional curvatures appearing in (4.3) to get

$$\tau_e = -\frac{\epsilon}{6} \sum_{l=1}^{3} \tilde{h}(A_{V_0} V_l, A_{V_0} V_l). \tag{4.4}$$

It is known that $A_{V_0} V_l$ is nothing but the horizontal component of $\tilde{\nabla}_{V_0} V_l$, where $\tilde{\nabla}$ denotes the Levi–Civita connection of $\tilde{h}$. We use standard computations involving (2.2) to obtain

$$A_{V_0} V_1 = -\frac{1}{2}(\tilde{h}(V_1, [iV_0, V_0])iV_0 + \tilde{h}(V_1, [jV_0, V_0])jV_0 + \tilde{h}(V_1, [kV_0, V_0])kV_0). \tag{4.5}$$

Similar equations are obtained for $A_{V_0} V_2$ and $A_{V_0} V_3$.

To compute the Lie brackets appearing in these formulae, we use the Levi–Civita connection $\bar{D}$ of the standard Riemannian metric $\bar{g}$ into $\mathbb{S}^7$. Therefore we apply (2.1) in (4.5) to get

$$\begin{cases} A_{V_0} V_1 = -\epsilon \bar{g}(V_0, V_0) iV_0, \\ A_{V_0} V_2 = -\epsilon \bar{g}(V_0, V_0) jV_0, \\ A_{V_0} V_3 = -\epsilon \bar{g}(V_0, V_0) kV_0. \end{cases} \tag{4.6}$$

Since V_0 is the horizontal lift of γ' to $(\mathbb{S}^7, \tilde{h})$, then $\bar{g}(V_0, V_0) = g(\gamma', \gamma') \circ \pi$. Therefore, if $\mathcal{U}\mathbb{S}^4$ denotes the unitary tangent bundle of $(\mathbb{S}^4, \frac{1}{v^2} \cdot h)$, we can define a smooth function $\psi \colon \mathcal{U}\mathbb{S}^4 \longrightarrow \mathbb{R}$ by $\psi(z) = g(z, z)$ (here z is a tangent vector at any point of $\mathbb{S}^4$ which has length one but relative to $\frac{1}{v^2}\cdot h$). Now we use this function and (4.6) in (4.4) to obtain

$$\tau_e = -\frac{\epsilon}{2}(\psi(\gamma'))^2 \circ \pi. \tag{4.7}$$

Consequently we have

$$\mathcal{W}(M_\gamma) = \int_{\gamma\times\mathbb{S}^3} \left\{ \left(\frac{1}{16}\kappa^2 + \frac{\epsilon}{2}(\psi(\gamma'))^2\right)^2 \circ \pi \right\} dsdA,$$

where dA also denotes the volume element of $(\mathbb{S}^3, d\sigma^2)$. Thus

$$\mathcal{W}(M_\gamma) = \frac{\mathrm{vol}(\mathbb{S}^3, d\sigma^2)}{256} \int_\gamma (\kappa^2 + \phi(\gamma'))^2 ds, \tag{4.8}$$

where $\phi(\gamma') = 8\epsilon(\psi(\gamma'))^2$.

The functional

$$\mathcal{G}(\gamma) = \int_\gamma (\kappa^2 + \phi(\gamma'))^2 ds$$

on closed curves γ appearing in (4.8) is defined as an *elastic energy* functional but with *potential* ϕ. It also appears naturally when one studies elasticity of curves in a submanifold of a Riemannian manifold ([14, 18]). We will call **ϕ-elasticae** the critical points of this functional ([8]).

Summing up, we have proved the following main theorem:

THEOREM 1: *Let $\bar{h} = \Phi(h, v, \epsilon)$ be a generalized Kaluza–Klein metric on $\mathbb{S}^7$ and $\mathcal{C}(\bar{h})$ its conformal class. If γ is a closed curve immersed into $\mathbb{S}^4$ and κ is its curvature function in $(\mathbb{S}^4, \frac{1}{v^2}\cdot h)$, then M_γ is a Willmore–Chen submanifold into $(\mathbb{S}^7, \mathcal{C}(\bar{h}))$ if and only if γ is a ϕ-elastica into $(\mathbb{S}^4, \frac{1}{v^2}\cdot h)$ with potential $\phi(\gamma') = 8\epsilon(g(\gamma', \gamma'))^2$.*

Remark 2: A similar result can be obtained if we use the standard Lorentzian metric $d\sigma_1^2$ on the fibre $\mathbb{S}^3$, instead of the Riemannian one (see Remark 1). So, a priori, one can obtain 4-dimensional Willmore–Chen submanifolds of any allowed index.

5. Critical points of $\mathcal{G}(\gamma) = \int_\gamma (\kappa^2 + \phi(\gamma'))^2 ds$.

Let $(N, \langle , \rangle)$ be a Riemannian manifold and denote by ∇ and R the Levi–Civita connection and the Riemannian curvature tensor, respectively. The classical notion of elastic curve involves the functional *total squared curvature* defined on the space of closed curves of a fixed length on $(N, \langle,\rangle)$. Although this concept is very old, there are more recent approaches (see, for instance, [15, 16, 17]). On the other hand, an *elastic energy with potential* functional appears when one studies the elasticity of curves living in a submanifold of $(N\langle,\rangle)$ (see [14]). In the last section, we have shown that this functional also appears for any metric on the 4-sphere, as a consequence of a reduction of variables for Willmore–Chen submanifolds, via the $\mathbb{S}^3$-symmetry of the fibration $\mathbb{S}^7(\mathbb{S}^4, \mathbb{S}^3)$.

Next, we are going to compute the Euler–Lagrange equations for this functional when the potential ϕ is the lifting to the unit tangent vector field $\mathcal{U}N$ of a function on N, although for our purposes it will be enough to consider $\mathcal{G}$ defined on the smooth manifold consisting of closed curves of a fixed length. One could vary $\mathcal{G}$ through curves which satisfy a given first-order boundary data.

Let $\gamma\colon I \subset \mathbb{R} \longrightarrow N$ be an arclength parametrized closed curve immersed in N with curvature function κ. Put $T = \gamma'$ and consider a variation $\Gamma \equiv \Gamma(s,t)$: $I \times (-\delta, \delta) \longrightarrow N$ of γ, $\Gamma(s,0) = \gamma(s)$, through closed curves. We use a standard terminology, which can be briefly reduced to: $V(s,t) = \partial\Gamma/\partial s$, $W(s,t) = \partial\Gamma/\partial t$, $v(s,t) = \langle V(s,t), V(s,t)\rangle^{\frac{1}{2}}$, $T(s,t) = \frac{1}{v}V(s,t)$ and $\kappa^2(s,t) = \langle\nabla_T T, \nabla_T T\rangle$. The following well-known lemma ([15]) collects some useful technical facts.

LEMMA 1: *Using the above notation, the following assertions hold:*

$$\begin{aligned}
[V, W] &= 0,\\
\frac{\partial v}{\partial t} &= \langle\nabla_T W, T\rangle v,\\
[[W,T],T] &= T(\langle\nabla_T W, T\rangle)T,\\
\frac{\partial \kappa^2}{\partial t} &= 2\langle\nabla_T^2 W, \nabla_T T\rangle - 4\langle\nabla_T W, T\rangle\kappa^2 + 2\langle R(W,T)T, \nabla_T T\rangle.
\end{aligned}$$

To calculate $\frac{\partial}{\partial t}\mathcal{G}(\Gamma(s,t))$ we combine the above lemma with a canonical argument based on integrations by parts. Since the boundary terms appearing in the computations drop out, we obtain from

$$\frac{\partial}{\partial t}\mathcal{G}(\Gamma(s,t))/_{t=0} = 0$$

the following Euler–Lagrange equation which characterizes the ϕ-elasticae of $(N, \langle,\rangle)$,

$$4(\kappa^2+\phi)\nabla_T^3 T + 8\frac{\partial}{\partial s}(\kappa^2+\phi)\nabla_T^2 T + (4\frac{\partial^2}{\partial s^2}(\kappa^2+\phi) + (\kappa^2+\phi)(7\kappa^2-\phi))\nabla_T T$$

$$+\frac{\partial}{\partial s}((\kappa^2+\phi)(7\kappa^2-\phi))T + 4(\kappa^2+\phi)R(T,\nabla_T T)T + 2(\kappa^2+\phi)\nabla\phi = 0, \tag{5.1}$$

where $\nabla\phi$ denotes the gradient of ϕ into $(N\langle,\rangle)$.

To better understand equation (5.1), we pose the following simple problem: *Let γ be a closed geodesic of a Riemannian manifold $(N,\langle,\rangle)$. The question is to characterize the smooth functions ϕ on N which make γ a ϕ-elastica.*

PROPOSITION 2: *Let γ be a closed geodesic of $(N,\langle,\rangle)$ and ϕ a smooth function on N. Then γ is a ϕ-elastica if and only if either:*
(1) *$\nabla\phi$ vanishes identically along γ, or*
(2) *γ is an integral curve of $\nabla\phi$.*

Proof: Since γ is geodesic, equation (5.1) becomes

$$\nabla\phi = T(\phi)T,$$

which proves the proposition.

Remark 3: It should be observed that when a potential ϕ is constant, then every geodesic is automatically a ϕ-elastica.

The following proposition exhibits a nice geometrical behaviour of the ϕ-elasticae, with ϕ being constant in a real-space form.

PROPOSITION 3: *Let $N^n(c)$ be an n-dimensional Riemannian manifold with constant sectional curvature c, and ϕ a constant. If γ is a ϕ-elastica in $N^n(c)$, then it lies in some $N^2(c)$ or $N^3(c)$ totally geodesic in $N^n(c)$.*

Proof: It is clear that $R(T,\nabla_T T)T = c\nabla_T T$. We use this fact in (5.1) and then we combine it with the Frenet equations of γ in $N^n(c)$. Finally, we apply a well-known argument due to Erbacher ([12]), to reduce codimension.

The reduction of codimension given in the last proposition does not work in general. In fact, we are going to show the existence of (closed) ϕ-elasticae in, for example, $\mathbb{S}^4(r)$ (the round 4-sphere of radius r), with three non-zero curvatures, that means lying fully in $\mathbb{S}^4(r)$. To do that, we assume $\mathbb{S}^3(1)$ is embedded totally umbilical (but non-totally geodesic) into $\mathbb{S}^4(r)$, namely $1 < r$. For a helix, we mean a regular curve which has all its curvatures constant. Now it is clear that an immersed curve γ in $S^3(1)$ is a helix if and only if it is a helix in $\mathbb{S}^4(r)$. We take a closed helix in $\mathbb{S}^3(1)$ ([5]), with curvature $\kappa \neq 0$ and torsion $\tau \neq 0$. Then γ is a helix in $\mathbb{S}^4(r)$ with non-zero curvatures $\bar\kappa$, $\bar\tau$ and $\bar\delta$. To determine a smooth function ϕ on $\mathbb{S}^4(r)$ making γ a ϕ-elastica, just make the following steps:

1. Define ϕ along γ and so we know $\partial\phi/\partial s$ and $\partial^2\phi/\partial s^2$.

2. Using the Euler–Lagrange equation, determine $\nabla\phi$ in terms of ϕ along γ and the geometry of γ. Therefore γ appears as a solution of that equation.

3. Finally, integrate to obtain ϕ locally around γ and extend it to $\mathbb{S}^4(r)$.

6. Some examples of Willmore–Chen submanifolds

First of all, we sketch the geometry of helices in a 3-sphere (see [5] for details). For the sake of simplicity we consider the round 3-sphere of radius one, $\mathbb{S}^3(1)$. Let p: $\mathbb{S}^3(1) \longrightarrow \mathbb{S}^2$ be the usual Hopf mapping which becomes a Riemannian submersion when the base is regarded as a round sphere $\mathbb{S}^2(\frac{1}{2})$ of radius $\frac{1}{2}$.

Let β: $\mathbb{R} \longrightarrow \mathbb{S}^2(\frac{1}{2})$ be an arclength parametrized curve with constant curvature $\rho \in \mathbb{R}$ in $\mathbb{S}^2(\frac{1}{2})$, and consider its Hopf tube $S_\beta = p^{-1}(\beta)$ (see [20]). Then S_β is a flat torus with constant mean curvature in $\mathbb{S}^3(1)$. Furthermore, S_β admits an obvious parametrization $Y(s,t)$ by means of fibres (s =constant) and horizontal lifts $\bar{\beta}$ of β (t = constant). If we choose a geodesic γ of S_β with *slope* $m \in \mathbb{R}$ (slope measured with respect to Y), then one can show that γ is a helix in $\mathbb{S}^3(1)$ with curvature κ and torsion τ given by

$$\kappa = \frac{\rho + 2m}{1+m^2} \quad \text{and} \quad \tau = \frac{1-\rho m - m^2}{1+m^2}. \tag{6.1}$$

This fact gives a characterization of the helices in $\mathbb{S}^3(1)$. In fact, given any helix γ in $\mathbb{S}^3(1)$ with curvatures (κ,τ) then it is a geodesic in the Hopf torus S_β with slope $m = (1-\tau)/\kappa$ where β has curvature $\rho = (\kappa^2+\tau^2+1)/\kappa$. Consequently, a helix in $\mathbb{S}^3(1)$ can be defined by the parameters (κ,τ) or, equivalently, from (ρ, m). In particular, a helix in $\mathbb{S}^3(1)$ is closed if and only if there exists a non-zero rational number, say q, such that

$$m = q\sqrt{\rho^2+4} - \frac{1}{2}\rho. \tag{6.2}$$

For convenience, we put π: $\mathbb{S}^7(2) \longrightarrow \mathbb{S}^4(1)$ as the Hopf fibration where $\mathbb{S}^7(2)$ and $\mathbb{S}^4(1)$ are assumed to be round spheres of radii 2 and 1, respectively. Now we have the canonical variation π_t: $(\mathbb{S}^7(2), \tilde{g}^t) \longrightarrow \mathbb{S}^4(1)$ of this Riemannian submersion. So we can apply Theorem 1 to these bundle-like metrics $\tilde{g}^t$ to obtain

COROLLARY 1: *Let γ be a closed curve immersed in $\mathbb{S}^4(1)$ with curvature function κ and $M_\gamma^t = \pi_t^{-1}(\gamma)$. Then M_γ^t is a Willmore–Chen submanifold in $(\mathbb{S}^7, \mathcal{C}(\tilde{g}^t))$ if and only if γ is a critical point for the functional $\int_\gamma(\kappa^2+2t^2)^2 ds$.*

Remark 4: Since the potential appearing in the last corollary is constant, namely $2t^2$, Proposition 3 implies that these ϕ-elasticae actually yield in $\mathbb{S}^2(1)$ or $\mathbb{S}^3(1)$ totally geodesic in $\mathbb{S}^4(1)$. We are going to take advantage of the simple geometry of helices into $\mathbb{S}^3(1)$ to construct examples of Willmore–Chen submanifolds in $(\mathbb{S}^7, \mathcal{C}(\tilde{g}^t))$ for certain values of the parameter t.

To better understand the next Theorem, we define for any real number $\lambda < 4$ a non-empty open subset $V(\lambda) \subset \mathbb{R}$ as follows:

$$V(\lambda) = \begin{cases} \left(\frac{\lambda-8-4\sqrt{4-\lambda}}{2\lambda}, \frac{\lambda-8+4\sqrt{4-\lambda}}{2\lambda}\right) \bigcup \left(\frac{8-\lambda-4\sqrt{4-\lambda}}{2\lambda}, \frac{8-\lambda+4\sqrt{4-\lambda}}{2\lambda}\right) & \text{if } \lambda \in (0,4), \\ \mathbb{R} - \{0\} & \text{if } \lambda = 0, \\ \left(\frac{8-\lambda+4\sqrt{4-\lambda}}{2\lambda}, \frac{8-\lambda-4\sqrt{4-\lambda}}{2\lambda}\right) \bigcup \left(\frac{\lambda-8+4\sqrt{4-\lambda}}{2\lambda}, \frac{\lambda-8-4\sqrt{4-\lambda}}{2\lambda}\right) & \text{if } \lambda < 0. \end{cases}$$

Then we have

THEOREM 2: *Let λ be a real number with $\lambda < 4$. For any rational number $q \in V(\lambda)$ there exists a closed helix $\gamma_{q,\lambda}$ in $\mathbb{S}^3(1)$ which is a critical point of the functional $\int_\gamma (\kappa^2 + \lambda)^2 ds$.*

Proof: First we use the Euler–Lagrange equation (5.1) with $\phi = \lambda =$ constant and R being the Riemannian curvature operator of $\mathbb{S}^3(1)$. Then a helix γ with curvatures (κ, τ) in $\mathbb{S}^3(1)$ is a solution if and only if

$$3\kappa^2 - 4\tau^2 + 4 - \lambda = 0. \tag{6.3}$$

Formula (6.3) can be written in terms of the parameters (ρ, m) as follows:

$$\lambda m^4 + 8\rho m^3 + (4\rho^2 + 2\lambda - 28)m^2 - 20\rho m - 3\rho^2 + \lambda = 0. \tag{6.4}$$

Now we combine (6.2) with (6.4) to see that a closed helix γ is a λ-elastica if and only if ρ and its rational slope q satisfy

$$\begin{aligned} F(\rho, q, \lambda) = & \left[\lambda q^4 + \left(\frac{3\lambda}{2} - 8\right)q^2 + \frac{1}{16}\right]\rho^4 + \left[(8-2\lambda)q^3 + \left(2 - \frac{\lambda}{2}\right)q\right]\rho^3\sqrt{\rho^2+4} \\ & + \left[8\lambda q^4 + (8\lambda - 60)q^2 + \frac{\lambda}{2}\right]\rho^2 + [(32-8\lambda)q^3 + (8-2\lambda)q]\rho\sqrt{\rho^2+4} \\ & + 16\lambda q^4 + (8\lambda - 112)q^2 + \lambda = 0. \end{aligned}$$

Now we undertake a straightforward long computation to show that for any rational number $q \in V(\lambda)$ one obtains a solution of the above equation.

COROLLARY 2: *For any real number $t \in (0, \sqrt{2})$ and any rational number $q \in V(2t^2)$, there exists a 4-dimensional Willmore–Chen submanifold with constant mean curvature in $(\mathbb{S}^7, \tilde{g}^t)$.*

COROLLARY 3: *For any positive real number t and any rational number $q \in V(-2t^2)$, there exists a 4-dimensional index three Willmore–Chen submanifold with constant mean curvature in $(\mathbb{S}^7, \tilde{g}_3^t)$, where $\{\tilde{g}_3^t\}$ denotes the canonical variation of the standard index three metric on $\mathbb{S}^7(2)$.*

Remark 5: It should be noticed that similar results could be obtained when we work with the canonical variation of the standard Lorentzian metric on $\mathbb{S}^7(2)$.

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